Lévy processes in free probability

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This is the continuation of a previous article that studied the relationship between the classes of infinitely divisible probability measures in classical and free probability, respectively, via the Bercovici–Pata bijection. Drawing on the results of the preceding article, the present paper outlines recent developments in the theory of Lévy processes in free probability.

The present article continues the account, begun in ref. 1, of some recent developments in the theory of free infinite divisibility and free Lévy processes.

Lévy processes are stochastic processes with independent and stationary increments. In general, processes with "independent" (but not necessarily stationary) increments, where "independent" can have a variety of meanings, are objects of wide current interest in stochastics (i.e., probability and statistics together) and mathematical physics. Not only are processes of this type of great interest in themselves, but they occur as important building blocks in other more structured processes. Ref. 2 contains state-of-the-art articles discussing a variety of aspects of this, but a number of the topics in question, however, are not treated there. Some of the further recent developments are presented in chapters 4 and 5 of ref. 3 and in refs. 4 and 5.

The present article outlines several results on processes with freely independent (and stationary) increments that are closely parallel to key results from the theory of Lévy processes in classical probability theory. In light of these and other related findings it seems certain that much more of interest can be done in studying the similarities, as well as the intriguing differences, between processes based on classical stochastic independence and free independence, respectively.

In *Lévy Processes in Free Probability*, we define free Lévy processes in complete analogy with the definition of classical Lévy processes. We show, subsequently, how the Bercovici–Pata bijection Λ studied in ref. 1 gives rise to a one-to-one (in law) correspondence between classical and free Lévy processes by virtue of its algebraic and topological properties, which were presented in ref. 1. In *Self-Decomposability and Free Stochastic Integration* we use the properties of Λ to construct certain stochastic integrals with respect to (w.r.t.) free Lévy processes, and we derive the free counterpart of the well known integral representation of classically self-decomposable random variables. We describe in *The Lévy–Itô Decomposition* a free version of the key Lévy–Itô decomposition of classical Lévy processes. Finally, in *Further Connections Between the Classical and Free Cases* a stochastic interpretation of the Bercovici–Pata bijection is given.

Throughout the present article, we make use, often without further comments, of the notations, definitions, etc. that were introduced in the preceding article (1).

1. Lévy Processes in Free Probability

In classical probability, Lévy processes form a very important area of research, both from the theoretical and applied points of view (see refs. 2 and 6–9). In free probability, such processes have already received quite a lot of attention (e.g. see refs. 10–12).

1.1. Definition: A *free Lévy process* (*in law*), affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, is a family $(Z_t)_{t\geq 0}$ of self-adjoint operators affiliated with $\mathcal{A}$, which satisfies the following conditions:

(*i*) whenever $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the increments

$$Z_{t_0}, Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \ldots, Z_{t_n} - Z_{t_{n-1}}$$

are freely independent operators;

(*ii*) $Z_0 = 0$;

(*iii*) for any s, t in $[0, \infty[$, the (spectral) distribution of $Z_{s+t} - Z_s$ does not depend on s; and

(*iv*) for any s in $[0, \infty[$, $Z_{s+t} \to Z_s$ *in probability*, as $t \to 0$, i.e. the (spectral) distributions $L\{Z_{s+t} - Z_s\}$ converge weakly to δ_0 (the Dirac measure at 0), as $t \to 0$.

A classical Lévy process in law is a family $(X_t)_{t\geq 0}$ of random variables on a probability space $(\Omega, \mathcal{F}, P)$, which satisfies conditions *i–iv* above except that free independence has to be replaced by classical independence in *i*. Such a process (X_t) is called a (genuine) Lévy process if, in addition, it satisfies the requirement that for almost all ω in Ω, the sample path $t \mapsto X_t(\omega)$ is right-continuous with left limits.

Let (Z_t) be a free Lévy process and let (μ_t) be the family of marginal distributions, i.e. $\mu_t = L\{Z_t\}$ for all t. As in the classical case, it is an immediate consequence of conditions *i* and *iii* that μ_t is $\boxplus$-infinitely divisible for all t. Note also that the following conditions are satisfied:

$$\mu_s \boxplus \mu_{t-s} = \mu_t, \quad (0 \leq s < t), \qquad \textbf{[1.1]}$$

and

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Abbreviation: w.r.t., with respect to.

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$$\mu_t \xrightarrow{w} \delta_0, \quad \text{as } t \searrow 0. \qquad [1.2]$$

Conversely, given any family (μ_t) of probability measures on $\mathbb{R}$, which satisfies Eqs. **1.1** and **1.2**, there exists a W^*-probability space $(\mathcal{A}, \tau)$ and a free Lévy process in law $(Z_t)_{t\geq 0}$ affiliated with $(\mathcal{A}, \tau)$ such that $L\{Z_t\} = \mu_t$ for all t. As noted in refs. 11 and 13, $(\mathcal{A}, \tau)$ can be constructed, loosely speaking, as the inductive limit of a directed system of free product von Neumann algebras. In classical probability, the corresponding existence result for classical Lévy processes in law follows by an application of Kolmogorov's consistency theorem. Because the Bercovici–Pata bijection preserves both conditions **1.1** and **1.2** (see *The Bercovici–Pata Bijection* in ref. 1), it follows then that we have the following correspondence between classical and free Lévy processes in law.

1.2. Proposition. *Let $(Z_t)_{t\geq 0}$ be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A}, \tau)$ and with marginal distributions (μ_t). Then there exists a (classical) Lévy process (in law) $(X_t)_{t\geq 0}$ [on some probability space $(\Omega, \mathcal{F}, P)$] with marginal distributions $[\Lambda^{-1}(\mu_t)]$.*

Conversely, for any (classical) Lévy process (in law) (X_t) with marginal distributions (μ_t), there exists a free Lévy process (in law) (Z_t) with marginal distributions $[\Lambda(\mu_t)]$.

2. Self-Decomposability and Free Stochastic Integration

In the preceding article (1), we briefly discussed, in *Infinite Divisibility, Self-Decomposability, and Stability* the notion of self-decomposability in classical probability. The following result, which was proved first by Wolfe (14) and later generalized and strengthened by Jurek and Verwaat (15), provides an alternative characterisation of self-decomposability: A random variable Y has law in the class $\mathcal{L}(*)$ of classically self-decomposable probability measures (see *Infinite Divisibility, Self-Decomposability, and Stability*) if and only if Y has a representation in the form

$$Y \overset{d}{=} \int_0^\infty e^{-t} dX_t, \qquad [2.1]$$

where X_t is a Lévy process satisfying

$$\mathbb{E}\{\log(1 + |X_1|)\} < \infty. \qquad [2.2]$$

The process $X = (X_t)_{t\geq 0}$ is termed the *background driving Lévy process* or the BDLP corresponding to Y; this is due to its role for processes of Ornstein–Uhlenbeck type (see ref. 16). The main purpose of this section is to outline the proof of a representation like **2.1** for any self-adjoint operator y with (spectral) distribution in the class $\mathcal{L}(\boxplus)$ of freely self-decomposable laws (as defined in *Infinite Divisibility, Self-Decomposability, and Stability*, of ref. 1).

The condition **2.2** is equivalent to asking that $\int_{\mathbb{R}\setminus[-1,1]} \log(1 + |s|)\rho_1(ds) < \infty$, where ρ_1 is the Lévy measure appearing in the generating triplet for $L\{X_1\}$ (see *The Bercovici–Pata Bijection* in ref. 1). Moreover, this condition is necessary and sufficient for the integrals $\int_0^R e^{-t} dX_t$ to converge, in probability, as $R \to \infty$; the limit being, by definition, the right-hand side of **2.1** (see ref. 15). The integrals $\int_0^R e^{-t} dX_t$, in turn, are defined as the limit, in probability, of Riemann sums

$$S_n = \sum_{j=1}^{n} e^{-t^{\#}_{n,j}} \cdot (X_{t_{n,j}} - X_{t_{n,j-1}}),$$

corresponding to subdivisions,

$$0 = t_{n,0} < t_{n,1} < t_{n,2} < \cdots < t_{n,n} = R, \quad t^{\#}_{n,j} \in [t_{n,j-1}, t_{n,j}], \quad (j = 1, 2, \ldots, n), \qquad [2.3]$$

subject to the condition that $\max\{t_{n,j} - t_{n,j-1} \mid j = 1, 2, \ldots, n\} \to 0$, as $n \to \infty$. Once again using the algebraic and topological properties of Λ (see *The Bercovici–Pata Bijection* in ref. 1), we derived (in ref. 17) the following free analog of the classical result described above.

2.1. Theorem. *Let y be a self-adjoint operator affiliated with a W^*-probability space $(\mathcal{A}, \tau)$. Then the distribution of y is $\boxplus$-self-decomposable if and only if y has a representation in law of the form*

$$y \overset{d}{=} \int_0^\infty e^{-t} dZ_t, \qquad [2.4]$$

for some free Lévy process (in law) (Z_t) affiliated with a W^*-probability space $(\mathcal{A}', \tau')$ and satisfying that $\int_{\mathbb{R}\setminus[-1,1]} \log(1 + |s|)\rho_1(ds) < \infty$, where ρ_1 is the Lévy measure appearing in the free generating triplet for $L\{Z_1\}$.

Proof (Sketch): Assume that $L\{y\} \in \mathcal{L}(\boxplus)$ and let Y be a classical random variable with distribution $\Lambda^{-1}(L\{y\})$. Then $L\{Y\} \in \mathcal{L}(*)$, and thus by the classical result described above, there is a classical Lévy process (X_t) such that Eqs. **2.2** and **2.1** are satisfied. Now choose a free Lévy process (Z_t) affiliated with some W^*-probability space $(\mathcal{A}', \tau')$ and corresponding to (X_t) as in *Proposition 1.2.* Then form the Riemann sums

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$$T_n = \sum_{j=1}^{n} e^{-t^{\#}_{n,j}}(Z_{t_{n,j}} - Z_{t_{n,j-1}}),$$

corresponding to subdivisions as in Eq. **2.3**. Let S_n denote the corresponding Riemann sums w.r.t. (X_t), and note then that because Λ preserves the affine structure on $\mathcal{ID}(*)$, we have that $L\{T_n\} = \Lambda(L\{S_n\})$ for all n. Hence, by continuity of Λ, $L\{T_n\} \xrightarrow{w} \Lambda(\int_0^R e^{-t}\, dX_t)$. We thus have established that the Riemann sums T_n converge in distribution. Note next that for n, m in $\mathbb{N}$, the difference $T_n - T_m$ can again be written as a Riemann-type sum corresponding to a certain subdivision. As above, it follows therefore that $L\{T_n - T_m\} = \Lambda(L\{S_n - S_m\})$, and we may conclude that (T_n) is a Cauchy sequence w.r.t. convergence in probability. Finally, one needs to call on the fact that the set of self-adjoint operators affiliated with $\mathcal{A}'$ is complete w.r.t. convergence in probability (cf. ref. 18). Consequently, there is a self-adjoint operator T affiliated with $\mathcal{A}'$ such that $T_n \to T$, in probability, as $n \to \infty$. It is easy to see that T does not depend on the particular choice of subdivisions and intermediate points, so we may define $\int_0^R e^{-t} dZ_t := T$. A similar argument applies to pass from the integrals $\int_0^R e^{-t} dZ_t$ to the limit $\int_0^\infty e^{-t} dZ_t$ as $R \to \infty$. By continuity of Λ, it follows then that

$$L\left\{\int_0^\infty e^{-t} dZ_t\right\} = \Lambda\left(L\left\{\int_0^\infty e^{-t} dX_t\right\}\right) = L\{y\}.$$

We refer to ref. 17 for further details.

3. The Lévy–Itô Decomposition

Historically, Lévy derived the Lévy–Khintchine representation of a measure μ in $\mathcal{ID}(*)$ by establishing first a decomposition of any (classical) Lévy process into two independent parts: a continuous part and a part that, loosely speaking, is the sum of the jumps of the process. This decomposition, now known as the Lévy–Itô decomposition, was later proved rigorously by Itô and is from the probabilistic viewpoint more basic than the Lévy–Khintchine representation. In order to describe precisely the sum of jumps of a Lévy process, one needs to introduce the concept of *Poisson random measures.* Before doing so, we recall that for any nonnegative number λ, the Poisson distribution P_λ with mean λ is the measure on the nonnegative integers, given by

$$P_\lambda(\{n\}) = \frac{\lambda^n}{n!} e^{-\lambda}, \quad (n \in \mathbb{N}_0).$$

3.1. Definition: Let $(\Theta, \mathcal{E}, \nu)$ be a σ-finite measure space. A Poisson random measure on $(\Theta, \mathcal{E}, \nu)$ is a collection $\{N(E) \mid E \in \mathcal{E}\}$ of random variables [defined on some probability space $(\Omega, \mathcal{F}, P)$] satisfying the following conditions:

(i) for each E in $\mathcal{E}$, $L\{N(E)\} = P_{\nu(E)}$;
(ii) if $E_1, \ldots, E_n$ are disjoint sets from $\mathcal{E}$, then $N(E_1), \ldots, N(E_n)$ are independent random variables; and
(iii) for each fixed ω in Ω, the mapping $E \mapsto N(E, \omega)$ is a measure on $\mathcal{E}$.

In case $\nu(E) = \infty$, condition i in the definition above means, by convention, that $N(E, \omega) = \infty$ for all ω in Ω. Recall next that a (standard) *Brownian motion* is a classical Lévy process (B_t) for which $L\{B_t\}$ is the Gaussian distribution with mean 0 and variance t. We then are ready to state the Lévy–Itô result mentioned above.

3.2. Theorem (Lévy–Itô). *Let (X_t) be a classical genuine Lévy process and let ρ be the Lévy measure appearing in the generating triplet for $L\{X_1\}$. Assume, for simplicity,*[¶] *that $\int_{-1}^{1} |x|\rho(dx) < \infty$. Then (X_t) has a representation in the form*

$$X_t \overset{\text{a.s.}}{=} \gamma t + \sqrt{a}B_t + \int_{]0,t] \times \mathbb{R}} x\, N(ds, dx), \qquad \textbf{[3.1]}$$

where $\gamma \in \mathbb{R}$, $a \geq 0$, (B_t) is a Brownian motion, and N is a Poisson random measure on $(]0, \infty[\times \mathbb{R}, \mathcal{B}(]0, \infty[\times \mathbb{R}), \text{Leb} \otimes \rho)$ (*here $\mathcal{B}$ denotes Borel σ-algebra and* Leb *denotes Lebesgue measure*). *Furthermore, the last two terms on the right-hand side of Eq.* **3.1** *are independent processes.*

The symbol $\overset{\text{a.s.}}{=}$ in Eq. **3.1** means that the two random variables are equal with probability 1 (“a.s.” stands for almost surely). The Poisson random measure N appearing in the right-hand side of Eq. **3.1** is specifically given by

$$N(E, \omega) = \#\{s \in]0, \infty[\mid (s, \Delta X_s(\omega)) \in E\},$$

for any Borel subset E of $]0, \infty[\times \mathbb{R}$ and where $\Delta X_s = X_s - \lim_{u \nearrow s} X_u$. Consequently, the integral in the right-hand side of Eq. **3.1** is indeed the sum of the jumps of X_s until time t: $\int_{]0,t]\times\mathbb{R}} x\, N(ds, dx) = \Sigma_{s\leq t}\, \Delta X_s$. The condition $\int_{-1}^{1} |x|\rho(dx) < \infty$ ensures that this sum converges.

3.3. Remark: Without the assumption $\int_{-1}^{1} |x|\rho(dx) < \infty$, one still has a Lévy–Itô decomposition, but it is slightly more complicated than Eq. **3.1**. In particular, the sum of jumps interpretation does not make sense, directly, in a rigorous fashion. We emphasize, though, that for applied purposes, the most interesting examples actually appear when the aforementioned condition is *not* satisfied.

[¶]Compare with *3.3 Remark* below.

In a forthcoming article (O.E.B.-N. and S.T., unpublished work), we prove a free analog of the Lévy–Itô decomposition, *Theorem 3.5* below. Before stating this result, we need to introduce the free counterparts to Brownian motion and Poisson random measures. A free Brownian motion is a free Lévy process with semicircular distributed increments. It corresponds, thus, to a classical Brownian motion via the correspondence described in *Proposition 1.2.* A free Poisson random measure is defined as follows.

3.4. Definition: Let $(\Theta, \mathcal{E}, \nu)$ be a σ-finite measure space, and put

$$\mathcal{E}_f = \{E \in \mathcal{E} \mid \nu(E) < \infty\}.$$

A free Poisson random measure on $(\Theta, \mathcal{E}, \nu)$ is a collection $\{M(E) \mid E \in \mathcal{E}_f\}$ of self-adjoint operators [affiliated with some W^*-probability space $(\mathcal{A}, \tau)$], satisfying the following conditions:

(*i*) for each E in $\mathcal{E}_f$, $L\{M(E)\} = \Lambda[P_{\nu(E)}]$, where Λ is the Bercovici–Pata bijection;
(*ii*) if $E_1, \ldots, E_n$ are disjoint sets from $\mathcal{E}_f$, then $M(E_1), \ldots, M(E_n)$ are freely independent; and
(*iii*) *if* $E_1, \ldots, E_n$ are disjoint sets from $\mathcal{E}_f$, then $M(\bigcup_{j=1}^n E_j) = \Sigma_{j=1}^n M(E_j)$.

The above definition of a free Poisson random measure may seem a little "poor" compared to that of a classical Poisson random measure. This definition, however, is sufficient to develop the integration theory needed to establish the free Lévy–Itô decomposition. Again for simplicity, we restrict attention to the case where $\int_{-1}^1 |x|\rho(dx) < \infty$ [the general case is discussed in a forthcoming article (O.E.B.-N. and S.T., unpublished work)].

3.5. Theorem. *Let (Z_t) be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, let ρ be the Lévy measure appearing in the free generating triplet for $L\{Z_1\}$ and assume that $\int_{-1}^1 |x|\rho(dx) < \infty$. Then (Z_t) has a representation in the form*

$$Z_t \overset{\mathrm{d}}{=} \gamma t + \sqrt{a}W_t + \int_{]0,t] \times \mathbb{R}} x\, M(ds, dx), \qquad \textbf{[3.2]}$$

where $\gamma \in \mathbb{R}$, $a \geq 0$, (W_t) is a free Brownian motion, and M is a free Poisson random measure on $(]0, \infty[\times \mathbb{R}, \mathcal{B}(]0, \infty[\times \mathbb{R}), \mathrm{Leb} \otimes \rho)$. Furthermore, the last two terms on the right-hand side of Eq. **3.2** *are freely independent processes, and the right-hand side of Eq.* **3.2** *is a free Lévy process.*

Proof (Sketch): Choose a classical Lévy process (X_t) corresponding to (Z_t) as in *Proposition 1.2.* Then ρ is the Lévy measure in the generating triplet for $L\{X_1\}$, and thus by *Theorem 3.2*, (X_t) has a representation in the form

$$X_t \overset{\mathrm{a.s.}}{=} \gamma t + \sqrt{a}B_t + \int_{]0,t] \times \mathbb{R}} x\, N(ds, dx),$$

where $\gamma \in \mathbb{R}$, $a \geq 0$, (B_t) is a Brownian motion, and N is a Poisson random measure on $(]0, \infty[\times \mathbb{R}, \mathcal{B}(]0, \infty[\times \mathbb{R}), \mathrm{Leb} \otimes \rho)$. Then consider a corresponding free Poisson random measure M on $(]0, \infty[\times \mathbb{R}, \mathcal{B}(]0, \infty[\times \mathbb{R}), \mathrm{Leb} \otimes \rho)$ affiliated with some W^*-probability space $(\mathcal{A}', \tau')$. We may assume, furthermore, that $(\mathcal{A}', \tau')$ contains a free Brownian motion (W_t), which is freely independent of M. The main part of the proof consists of establishing a theory for integration w.r.t. M, such that the integral $\int_{]0,t]\times\mathbb{R}} x\, M(ds, dx)$ is realized as a self-adjoint operator affiliated with $(\mathcal{A}', \tau')$ and satisfying that

$$L\left\{\int_{]0,t] \times \mathbb{R}} x\, M(ds, dx)\right\} = \Lambda\left(L\left\{\int_{]0,t] \times \mathbb{R}} x\, N(ds, dx)\right\}\right). \qquad \textbf{[3.3]}$$

This is obtained by virtue of the bijection Λ in much the same way as the integrals w.r.t. free Lévy processes in *Theorem 2.1* were constructed. Once Eq. **3.3** has been established, it follows that

$$\begin{aligned} L\left\{\gamma t + \sqrt{a}W_t + \int_{]0,t]\times\mathbb{R}} x\, M(ds, dx)\right\} &= \delta_{\gamma t} \boxplus D_{\sqrt{a}}L\{W_t\} \boxplus L\left\{\int_{]0,t]\times\mathbb{R}} x\, M(ds, dx)\right\} \\ &= \Lambda(\delta_{\gamma t}) \boxplus D_{\sqrt{a}}\Lambda(L\{B_t\}) \boxplus \Lambda\left(L\left\{\int_{]0,t]\times\mathbb{R}} x\, N(ds, dx)\right\}\right) \\ &= \Lambda\left(\delta_{\gamma t} * D_{\sqrt{a}}L\{B_t\} * L\left\{\int_{]0,t]\times\mathbb{R}} x\, N(ds, dx)\right\}\right) \\ &= \Lambda\left(L\left\{\gamma t + \sqrt{a}B_t + \int_{]0,t]\times\mathbb{R}} x\, N(ds, dx)\right\}\right) \\ &= \Lambda(L\{X_t\}) \\ &= L\{Z_t\}, \end{aligned}$$

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which proves Eq. **3.2**. Now Eq. **3.2** only means that for each t, the two appearing operators have the same (spectral) distribution. Therefore, to conclude the proof, one has to verify, furthermore, that the right-hand side of Eq. **3.2** is indeed a free Lévy process (in law) [further details will be provided in an upcoming article (O.E.B.-N. and S.T., unpublished work)].

4. Further Connections Between the Classical and Free Cases

In another forthcoming article (O.E.B.-N. and S.T., unpublished work), we establish a further connection between the free and classical settings. We show that there exists a one-to-one mapping $\Upsilon : \mathcal{ID}(*) \to \mathcal{ID}(*)$ such that, for any μ in $\mathcal{ID}(*)$, the free cumulant transform $\mathcal{C}_{\Lambda(\mu)}(z)$ of $\Lambda(\mu)$ is equal to the classical cumulant transform $C_\lambda(\zeta)$ of the probability distribution $\lambda = \Upsilon(\mu)$, when $z = i\zeta$ and $\zeta < 0$. This mapping has algebraic and topological properties similar to those of Λ. Furthermore, the law λ is identifiable as that of a certain stochastic integral with respect to the (classical) Lévy process Y_t, for which the law of Y_1 is equal to μ.

More specifically, the mapping Υ is defined as follows. For any μ in $\mathcal{ID}(*)$, with generating triplet (a, ρ, η), we take $\lambda = \Upsilon(\mu)$ to be the element of $\mathcal{ID}(*)$, the generating triplet of which is $(2a, \sigma, \eta')$, where

$$\eta' = \eta + \int_0^\infty \int_{-\infty}^\infty \{1_{[-1,1]}(t) - 1_{[-1,1]}(x^{-1}t)\} t\rho(x^{-1}dt)e^{-x}dx$$

and

$$\sigma(dt) = \int_0^\infty \rho(x^{-1}dt)e^{-x}dx. \qquad \textbf{[4.1]}$$

This mapping has the properties that $\Upsilon[\mathcal{S}(*)] \subseteq \mathcal{S}(*)$ and $\Upsilon[\mathcal{L}(*)] \subseteq \mathcal{L}(*)$. Furthermore, for any measure $\mu \in \mathcal{ID}(*)$ we have

$$\mathcal{C}_{\Lambda(\mu)}(i\zeta) = \int_0^\infty e^{-x} C_\mu(\zeta x)dx = C_{\Upsilon(\mu)}(\zeta). \qquad \textbf{[4.2]}$$

Now, let Y_t be a classical Lévy process such that $L\{Y_1\} = \mu$, and define a new random variable X by

$$X = -\int_0^1 \log(1-t)dY_t. \qquad \textbf{[4.3]}$$

Then

$$C_{L\{X\}}(\zeta) = \int_0^\infty e^{-x} C_\mu(\zeta x)dx. \qquad \textbf{[4.4]}$$

In other words, comparing to Eq. **4.2**, one sees that the free cumulant transform $\mathcal{C}_{\Lambda(\mu)}(z)$ for $z = i\zeta$ ($\zeta < 0$), is equal to the classical cumulant transform of the random variable X, which is given as the simple integral **4.3** with respect to the Lévy process Y_t generated by μ.

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1. Barndorff-Nielsen, O. E. and Thorbjørnsen, S. (2002) *Proc. Natl. Acad. Sci. USA* **99,** 16568–16575.
2. Barndorff-Nielsen, O. E., Mikosch, T. & Resnick, S., eds. (2001) *Lévy Processes—Theory and Applications* (Birkhäuser, Boston).
3. Holevo, A. S. (2001) *Statistical Structure of Quantum Theory* (Springer, Heidelberg).
4. Muraki, N. (2002) *Inf. Dim. Anal. Quantum Prob. Rel. Top.* **5,** 113–134.
5. Ben Ghorbal, A. & Schürmann, M. (2001) *On the Algebraic Foundations of Non-Commutative Probability Theory* (Univ. of Greifswald, Greifswald, Germany).
6. Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions* (Cambridge Univ. Press, Cambridge, U.K.), Vol. 68.
7. Bertoin, J. (1997) in *Lectures on Probability Theory and Statistics,* ed. Bernard, P. (Springer, Berlin), pp. 4–91.
8. Bertoin, J. (2000) *Subordinators, Lévy Processes with No Negative Jumps and Branching Processes* (Univ. of Aarhus, Aarhus, Denmark).
9. Le Gall, J.-F. (1999) *Spatial Branching Processes, Random Snakes and Partial Differential Equations* (Birkhäuser, Basel).
10. Anshelevich, M. (2002) *J. Funct. Anal.* **188,** 292–315.
11. Biane, P. (1998) *Math. Z.* **227,** 143–174.
12. Biane, P. & Speicher, R. (1998) *Probab. Theor. Rel. Fields* **112,** 373–409.
13. Voiculescu, D. V. (2000) in *Lectures on Probability Theory and Statistics* (Springer, Berlin), pp. 279–349.
14. Wolfe, S. J. (1982) *Stochastic Process. Appl.* **12,** 301–312.
15. Jurek, Z. J. & Verwaat, W. (1983) *Z. Wahrscheinlichkeitstheorie verw. Geb.* **62,** 247–262.
16. Barndorff-Nielsen, O. E. (1998) *Finance Stochastics* **2,** 41–68.
17. Barndorff-Nielsen, O. E. & Thorbjørnsen, S. (2002) *Bernoulli* **8,** 323–366.
18. Nelson, E. (1974) *J. Funct. Anal.* **15,** 103–116.